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RESUMMED $B \to X_u \, l \, \nu$ DECAY DISTRIBUTIONS TO NEXT-TO-LEADING ORDER

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Abstract

We perform factorization of the most general distribution in semileptonic $B \to X_u$ decays and we resum the threshold logarithms to next-to-leading order. From this (triple-differential) distribution, any other distribution is obtained by integration. As an application of our method, we derive simple analytical expressions for a few distributions, resummed to leading approximation. It is shown that the shape function can be directly determined by measuring the distribution in m_X^2/E_X^2, not in m_X^2/m_B^2. We compute the resummed hadron energy spectrum, which has a "Sudakov shoulder", and we show how the distribution in the singular region is related to the shape function. We also present an improved formula for the photon spectrum in $B \to X_s \, \gamma$, which includes soft-gluon resummation and non-leading operators in the effective hamiltonian. We explicitly show that the same non-perturbative function — namely the shape function — controls the non-perturbative effects in all the distributions in the semileptonic and in the rare decay.

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1 Introduction

In this note we discuss factorization and threshold resummation to next-to-leading order (NLO) in semi-inclusive B decays:

$$B \to X_q + (\text{non QCD partons}), \tag{1}$$

where X_q is any hadronic final state containing the light quark $q = u, s$ coming from b fragmentation. The non-QCD partons are a lepton–neutrino pair or a photon.

In sec. 2 we consider the most general distribution in the decay

$$B \to X_u + l + \nu. \tag{2}$$

A general expression for the triple differential distribution is presented, from which any other distribution is obtained by integration. The process (2) is characterized by three energy scales:

$$m_B, \quad Q, \quad m_X, \tag{3}$$

where[1]

$$Q \equiv 2E_X. \tag{4}$$

Factorization is achieved with two different steps, according to the method developed in [1]:

1. We factorize heavy mass effects by taking the limit

   $$m_B \to \infty, \qquad Q, \quad m_X \to \text{const.} \tag{5}$$

   The distributions contain logarithms of the heavy mass, $\log m_B/Q$, related to the non-conservation of the transition currents in the static limit for the beauty quark [2]. This step is discussed in sec. 2.1.

2. We factorize the large infrared logarithms that appear in the semi-inclusive region,

   $$\alpha_S^n\, \mathcal{D}_k(z) \qquad (0 \le k \le 2n-1), \tag{6}$$

   by taking the limit

   $$z \to 1 \tag{7}$$

   where

   $$z \equiv 1 - \frac{m_X^2}{Q^2}. \tag{8}$$

   We have defined:

   $$\mathcal{D}_k(z) \equiv \left[\frac{\log^k(1-z)}{1-z}\right]_+ . \tag{9}$$

   Plus-distributions are defined as usual as $P(z)_+ \equiv P(z) - \delta(1-z)\int_0^1 dx\, P(x)$. This step is discussed in sec. 2.2.

[1]The factor 2 multiplying E_X is inserted for convenience.

The basic idea of our approach is to use the kinematical variables

$$w \equiv \frac{Q}{m_B} \quad (0 \leq w \leq 2) \qquad \text{and} \qquad z, \tag{10}$$

the latter replacing the generally used one

$$u \equiv 1 - \frac{m_X^2}{m_B^2}. \tag{11}$$

In other words, we consider (two times) the hadronic energy Q as the hard scale of the process, rather than the b quark mass m_b (or, equivalently, the B-meson mass). The reason for this choice is that the infrared structure of the decay is not modified by the limit (5): this implies that m_B cannot be the relevant hard scale; the heavy mass acts only as an energy resevoir for the light partons in the final state and has not any fundamental dynamical meaning. Using the variable z instead of u largely simplifies the logarithmic structure. In the latter variable, the decay distributions contain, at one loop, terms of the form [3]:

$$\mathcal{D}_1(u), \qquad \log w\, \mathcal{D}_0(u), \qquad \mathcal{D}_0(u), \qquad \log^2 w\, \delta(1-u), \qquad \log w\, \delta(1-u), \tag{12}$$

while in the variable z one has only terms of the form:

$$\mathcal{D}_1(z), \qquad \mathcal{D}_0(z), \qquad \log w\, \delta(1-z). \tag{13}$$

In the latter case, the two different logarithmic structures basically decouple.

We explicitly show that a unique function $f(z)$, depending only on z, factorizes the long-distance effects in any distribution in the decay (2). The perturbative expansion of $f(z)$ reads:

$$f(z) \equiv \delta(1-z) - \alpha_S A_1 \mathcal{D}_1(z) + \alpha_S B_1 \mathcal{D}_0(z) + O(\alpha_S^2), \tag{14}$$

where the constants multiplying the distributions are given by:

$$A_1 = \frac{C_F}{\pi}, \qquad B_1 = -\frac{7}{4}\frac{C_F}{\pi} \tag{15}$$

and $C_F = (N_c^2 - 1)/(2N_c) = 4/3$. This function is related, by a short-distance coefficient function C, to the shape function in the effective theory [4, 5], defined as

$$\varphi(k_+) \equiv \langle B(v)| h_v^\dagger\, \delta(k_+ - iD_+)\, h_v |B(v)\rangle, \tag{16}$$

where

$$k_+ \equiv -\frac{m_X^2}{Q}. \tag{17}$$

C is defined by the relation [6, 7]:

$$\varphi(k_+; Q)^{QCD} = \int C\left(k_+ - k'_+; Q, \mu\right)\, \varphi\left(k'_+; \mu\right)\, dk'_+, \tag{18}$$

where we have defined

$$\varphi(k_+; Q)^{QCD} = \frac{1}{Q} f(z) \tag{19}$$

and $\mu < Q$ is the ultraviolet cut-off or renormalization point of the effective theory. The shape function factorizes the long-distance effects — both perturbative and non-perturbative — in the process up to the scale μ. The relation between the QCD and the effective theory variable is

$$z = 1 + \frac{k_+}{Q}. \tag{20}$$

The relevance of the variable z to describe the long-distance effects — perturbative and non-perturbative — in (1) is also supported by the following argument. From dimensional analysis, the shape function is of the form

$$\varphi(k_+;\mu) = \frac{1}{\mu}\,\psi\left(\frac{k_+}{\mu}\right). \tag{21}$$

The coefficient function in eq. (18) does not contains large logarithms of the ratio Q/μ if we choose

$$\mu \sim O(Q), \tag{22}$$

implying that

$$\frac{k_+}{\mu} \sim -\frac{m_X^2}{Q^2} = -1 + z. \tag{23}$$

The last equation implies that long-distance effects are described by a function of the variable z, Q.E.D.. Let us stress that $f(z)$ "includes" the shape function but does not coincide with the latter, because it also contains some short-distance effects (soft gluons with energies bewteen μ and Q and hard collinear contributions) [6, 7].

The shape function was originally derived with an Operator Product Expansion (OPE) and was claimed to be the universal non-perturbative component in (1) since the early papers on the subject [4, 5]: we present here an explicit proof of this. On the perturbative side, our analysis can be considered as the resummed analogue of the distributions computed to $O(\alpha_S)$ in [3]; some of the decay distributions presented in [3] were computed before in [8, 9, 10] [2].

In the second part of sec. 2 we present simple analytical expressions for a few distributions in (2), resummed to leading logarithmic accuracy. An interesting distribution from the theoretical side is the hadron energy spectrum. This distribution has a singularity in fixed-order perturbation theory inside the physical region, in the point [3]

$$E_X = \frac{m_B}{2}. \tag{24}$$

This phenomenon originates from the mismatch of the allowed phase space in lowest order in α_S:[3] $E_X \le m_B/2$, and in higher orders: $E_X \le m_B$. The all-order resummation eliminates the singularity and produces a characteristic behaviour known as the "Sudakov shoulder" [12]. Actually, a singularity very close to point (24) remains even after soft-gluon resummation. The latter has a different origin: it is related to the Landau pole in the running coupling and is factorized with the introduction of the shape function.

In sec. 3 we present an improved formula for the photon spectrum in the rare decay

$$B \to X_s + \gamma \tag{25}$$

in the threshold region. We also explicitly show that the same function, the effective form factor $f(z)$, factorizes all the long-distance effects in processes (2) and (25).

The distributions in (2) and (25) can be roughly divided into two classes. The first set contains distributions *not* involving integration over the hadronic energy. A first example is the z distribution in (2):

$$\frac{1}{\Gamma_0}\frac{d\Gamma_{SL}}{dz} = \delta(1-z) - \frac{\alpha_S C_F}{\pi}\,\mathcal{D}_1(z) - \frac{7}{4}\frac{\alpha_S C_F}{\pi}\,\mathcal{D}_0(z) + \cdots, \tag{26}$$

where $\Gamma_0 = G_F^2\, m_b^5 |V_{ub}|^2/\left(192\pi^3\right)$ is the total semileptonic width in Born approximation. A second example is the photon spectrum in the rare decay:

$$\frac{1}{\Gamma_{RD}}\frac{d\Gamma_{RD}}{dx_\gamma} = \delta(1-x_\gamma) - \frac{\alpha_S C_F}{\pi}\mathcal{D}_1(x_\gamma) - \frac{7}{4}\frac{\alpha_S C_F}{\pi}\,\mathcal{D}_0(x_\gamma) + \cdots, \tag{27}$$

[2] Analogous computations were performed a long time ago in the context of one-loop QED corrections to μ decay [11].

[3] Kinematics involve the decay of a massive particle into three massless particles and it is analogous to the well-known case $e^+e^- \to \gamma^*, Z^* \to q\bar{q}g$.

where Γ_{RD} is the total $b \to s\gamma$ width and

$$x_\gamma \equiv \frac{2E_\gamma}{m_B} \qquad (0 \le x_\gamma \le 1)\,. \tag{28}$$

In the decay (25), the hadronic energy is never integrated because kinematics fixes $Q \approx m_B$ (see sec. 3 for a proof). As expected, the leading terms, $\mathcal{D}_1(z)$ and $\mathcal{D}_1(x_\gamma)$, have the same coefficient. It is instead non trivial that the subleading terms also, $\mathcal{D}_0(z)$ and $\mathcal{D}_0(x_\gamma)$, have the same coefficient, $-7/4$, and that these distributions have a perturbative expansion similar to the one for $f(z)$ (cf. eq. (14)). In general, by measuring distributions in this class, one can directly determine the effective form factor $f(z)$, or equivalently, the shape function $\varphi(k_+)$.

The second class contains distributions in which the hadronic energy is integrated over. As examples, consider the hadron-mass distribution [9] in (2)

$$\frac{1}{\Gamma_0}\frac{d\Gamma_{SL}}{du} = \delta(1-u) - \frac{\alpha_S C_F}{\pi}\mathcal{D}_1(u) - \frac{31}{12}\frac{\alpha_S C_F}{\pi}\mathcal{D}_0(u) + \cdots\,, \tag{29}$$

or the electron spectrum [8] in (2),

$$\frac{1}{2\Gamma_0}\frac{d\Gamma_{SL}}{dx_e} = 1 - \frac{\alpha_S C_F}{2\pi}\log^2(1-x_e) - \frac{31}{12}\frac{\alpha_S C_F}{\pi}\log(1-x_e) + \cdots\,, \tag{30}$$

where

$$x_e \equiv \frac{2E_e}{m_B} \qquad (0 \le x_e \le 1)\,. \tag{31}$$

The distributions (29) and (30) have the same leading behaviour as in (26) and (27); the coefficient of the subleading terms, $-31/12$, is instead different because the integration over the hadronic energy affects them [13]. The distributions in this class are not directly related to the shape function and the extraction of the latter from the experimental data requires in general a deconvolution.

Finally, in sec. 4 we present our conclusions and an outlook of future developments.

2 Semileptonic decay

Let us consider the hadronic tensor containing *all* QCD dynamics:

$$W_{\mu\nu} \equiv \sum_{X_u} \langle B|J_\nu^+|X_u\rangle\,\langle X_u|J_\mu|B\rangle\;(2\pi)^3\,\delta^4(p_B - q - p_X), \tag{32}$$

where q is the momentum of the lepton–neutrino pair and $J_\mu(x) = \overline{u}(x)\gamma_\mu(1-\gamma_5)\,b(x)$ is the $b \to u$ current of the Standard Model. The latter involves five independent form factors. We find it convenient to use a modified parametrization with respect to the one proposed in [3]:

$$\begin{aligned} W_{\mu\nu}(p_B;p_X) \;=\; \frac{1}{2v\cdot p_X}\Big[&\left(n_\mu v_\nu + n_\nu v_\mu - g_{\mu\nu}\,v\cdot n - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\right)W_1(\varsigma,w) - g_{\mu\nu}\,W_2(\varsigma,w) + \\ &+ v_\mu v_\nu\,W_3(\varsigma,w) + (n_\mu v_\nu + n_\nu v_\mu)\,W_4(\varsigma,w) + n_\mu n_\nu W_5(\varsigma,w)\Big]\,, \end{aligned} \tag{33}$$

where

$$v^\mu \equiv \frac{p_B^\mu}{m_B} = (1;0,0,0) \tag{34}$$

is the velocity of the beauty meson, which we take at rest, while

$$n^{\mu} \equiv \frac{p_X^{\mu}}{v \cdot p_X} = (1; 0, 0, -\sqrt{\varsigma}) \tag{35}$$

is the normalized momentum of the jet, which we have taken in the minus direction. We have defined:

$$\zeta \equiv 1 - \frac{m_X^2}{E_X^2} \qquad\qquad (0 \le \zeta \le 1)\,. \tag{36}$$

Note that $1 - \varsigma = 4\,(1 - z)$. We have inserted for convenience a factor $1/(2v \cdot p_X)$ multiplying all the form factors. Note that $n \cdot v = 1$ and that $n^2 = 1 - \varsigma \ll 1$ for $\varsigma \lesssim 1$, i.e. n is close to the light cone in the threshold region.

The form factors can be decomposed as:

$$W_i(z; w; \alpha_S) = \delta_{i1}\ \delta(1-z)\,\theta\,(1-w) + \delta_{i1}\frac{\alpha_S C_F}{\pi}\,s\,(z)\ + \frac{\alpha_S C_F}{\pi}\delta(1-z) v_i\,(w) + \frac{\alpha_S C_F}{\pi}\,r_i\,(\zeta; w) + O(\alpha_S^2). \tag{37}$$

The function $s\,(z)$ contains the plus distributions, i.e. the long-distance contributions:

$$s\,(z) = -\mathcal{D}_1\,(z) - \frac{7}{4}\mathcal{D}_0\,(z)\,. \tag{38}$$

The functions $v_i\,(w)$ come from virtual effects — they are proportional to $\delta(1-z)$ — while the functions $r_i\,(\zeta; w)$ originate from real emission only. The explicit expressions for these functions can be extracted from [3] and are reported in the following. The "virtual" functions read:

$$\begin{aligned}
v_1\,(w) &= -\frac{3}{2}\log w - \mathrm{Li}_2\,(1-w) - \frac{w \log w}{2\,(1-w)} - \frac{5}{4} - \frac{\pi^2}{3};\\
v_2\,(w) &= 0;\\
v_3\,(w) &= 0;\\
v_4\,(w) &= \frac{1}{2\,(1-w)}\left(\frac{w \log w}{1-w} + 1\right);\\
v_5\,(w) &= \frac{w}{2\,(1-w)}\left(\frac{1-2w}{1-w}\log w - 1\right),
\end{aligned} \tag{39}$$

where $\mathrm{Li}_2\,(z) \equiv \sum_{n=1}^{\infty} z^n/n^2 \quad (|z| \le 1)$ is the standard dilogarithm. The "real" functions are:

$$\begin{aligned}
r_1\,(\zeta; w) &= \frac{w^2}{4} - \frac{(8-w)\,(2-w)}{4\zeta} + \left[\frac{w\,(2-w)}{8} + \frac{(8-w)\,(2-w)}{8\varsigma}\right] H\,(\varsigma) + \frac{1}{1-z}\left[H\,(\varsigma) + \log\,(1-z)\right];\\
r_2\,(\varsigma, w) &= \frac{w\,(8-w)}{8} + \frac{32 - 8w + w^2}{8\varsigma} - \frac{H\,(\varsigma)}{16}\left[w^2\zeta + 2w\,(4-w) + \frac{32 - 8w + w^2}{\varsigma}\right];\\
r_3\,(\varsigma, w) &= -\frac{w\,(8-3w)}{8} + \frac{32 + 22w - 3w^2}{4\varsigma} - \frac{3w\,(12-w)}{8\varsigma^2} +\\
&\quad + \frac{H\,(\varsigma)}{16}\left[w^2\zeta + 5w\,(4-w) - \frac{64 + 56w - 7w^2}{\varsigma} + \frac{3w\,(12-w)}{\varsigma^2}\right];\\
r_4\,(\varsigma, w) &= -\frac{w^2}{4} - \frac{w\,(32-5w)}{8\varsigma} + \frac{3w\,(12-w)}{8\varsigma^2} - \frac{wH\,(\varsigma)}{16}\left[8 - 3w - \frac{2\,(22-3w)}{\varsigma} + \frac{3\,(12-w)}{\varsigma^2}\right];\\
r_5\,(\varsigma, w) &= -\frac{w\,(8+w)}{8\varsigma} - \frac{3w\,(12-w)}{8\varsigma^2} + \frac{wH\,(\varsigma)}{16}\left[w - \frac{2\,(2-w)}{\varsigma} + \frac{3\,(12-w)}{\varsigma^2}\right].
\end{aligned} \tag{40}$$

We have defined:

$$H(\varsigma) \equiv \frac{1}{\sqrt{\zeta}} \log \frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}. \tag{41}$$

In the semi-inclusive region:

$$H(\varsigma) = -\log(1-z) + O(1-z), \tag{42}$$

implying that there is no $1/(1-z)$ singularity in r_1 for $z \to 1$. The functions r_i have at most a $\log(1-z)$ singularity for $z \to 1$, which gives no logarithmic enhancement after integration over z. Another property is that the "real" functions do not have any singularity of the form $1/\zeta^2$ for $\zeta \to 0$, but at most a singularity of the form $1/\zeta$ (the point $\varsigma = 0$ corresponds to the final hadronic system at rest). As already noted, the Born term in eq. (37) has the additional restriction $w \leq 1$.

2.1 Heavy mass effects

Let us now consider the limit of infinite mass for the beauty quark, keeping the other kinematical invariants fixed, i.e. the limit (5). In terms of our variables, this is:

$$w \to 0, \quad \zeta \to \text{const.} \tag{43}$$

In the limit (43) the "virtual" functions behave as:

$$\begin{aligned} v_1 &\to -\frac{3}{2}\log w; \\ v_2 &\to 0; \\ v_3 &\to 0; \\ v_4 &\to \frac{1}{2}; \\ v_5 &\to 0. \end{aligned} \tag{44}$$

The "real" functions have limits:

$$\begin{aligned} r_1 &\to -\frac{4}{\zeta} + \frac{2}{\zeta}H(\varsigma) + \frac{1}{1-z}\left[H(\varsigma) + \log(1-z)\right]; \\ r_2 &\to \frac{4}{\zeta} - \frac{2}{\zeta}H(\varsigma); \\ r_3 &\to \frac{8}{\zeta} - \frac{4}{\zeta}H(\varsigma); \\ r_4 &\to 0; \\ r_5 &\to 0. \end{aligned} \tag{45}$$

Only the function v_1 diverges logarithmically in this limit: all the other functions have a finite limit or vanish. The "real" functions are finite because real emission diagrams do not generate logarithms of the heavy mass, so $\log w$ does not appear. The logarithmic divergence of v_1 is associated to the term in W_1 :

$$W_1 = \delta(1-z)\left[1 - \frac{3}{2}\frac{\alpha_S C_F}{\pi}\log w\right] + \cdots. \tag{46}$$

Equation (46) can be understood by considering the general properties of the vector and axial current containing a heavy and a light field. The matrix element of these current between quark states in QCD are of the form

[14]:

$$\langle u|V_\mu|b\rangle = \overline{u}_u \left\{ \left[1 + \frac{\alpha_S C_F}{\pi} \left(\frac{3}{4} \log m_B + \cdots \right) \right] \gamma_\mu + \frac{\alpha_S C_F}{2\pi} v_\mu \right\} u_b \tag{47}$$

and

$$\langle u|A_\mu|b\rangle = \overline{u}_u \left\{ \left[1 + \frac{\alpha_S C_F}{\pi} \left(\frac{3}{4} \log m_B + \cdots \right) \right] \gamma_\mu \gamma_5 - \frac{\alpha_S C_F}{2\pi} v_\mu \gamma_5 \right\} u_b, \tag{48}$$

where the dots denote terms dependent on the kinematics of the external states. Substituting the curly brackets in the above equations in place of the currents in eq. (32), we recover the logarithmic term in eq. (46) for W_1, together with the delta-function contribution in W_4.

The appearance of the logarithm of the heavy mass in the matrix elements (47) and (48) can be understood by considering the effective theory: it is related to the non-conservation of the vector and axial currents in the static limit

$$m_b \to \infty. \tag{49}$$

The currents are multiplicatively renormalized in the effective theory [2] and their matrix elements between quark states read:

$$\begin{aligned} \langle u|\widetilde{V}_\nu\left(\mu\right)|b\rangle &= \overline{u}_u \left[1 + \frac{\alpha_S C_F}{\pi} \left(\frac{3}{4} \log \mu + \cdots \right) \right] \gamma_\nu u_b, \\ \langle u|\widetilde{A}_\nu\left(\mu\right)|b\rangle &= \overline{u}_u \left[1 + \frac{\alpha_S C_F}{\pi} \left(\frac{3}{4} \log \mu + \cdots \right) \right] \gamma_\nu \gamma_5 u_b, \end{aligned} \tag{50}$$

where μ is the renormalization point and the dots denote μ-independent terms. As expected, the logarithmic dependence on the renormalization point μ of the effective currents matches the dependence on the heavy mass m_b in the matrix elements of the full QCD currents.

2.2 Infrared factorization

To perform factorization of infrared logarithms, the form factors are conveniently written as:

$$W_i(z;w;\alpha_S) = \delta_{i1}\, f\left(z;\alpha_S\right) + \frac{\alpha_S C_F}{\pi}\, \delta(1-z)\, v_i\left(w\right) + \frac{\alpha_S C_F}{\pi}\, r_i\left(\zeta;w\right) - \delta_{i1}\delta(1-z)\theta\left(w-1\right) + O(\alpha_S^2). \tag{51}$$

The function $f\left(z\right)$, defined previously (eq. (14)), contains the long-distance effects. The soft-gluon resummation of $f(z)$ allows us to describe the semi-inclusive region

$$1 - z \ll 1 \tag{52}$$

and has been performed to NLO in [10, 15, 7]; $f(z)$ is perturbatively computable as long as

$$1 - z \gg \frac{\Lambda}{Q}, \tag{53}$$

where Λ is the QCD scale. In the region

$$1 - z \sim \frac{\Lambda}{Q} \tag{54}$$

$f(z)$ acquires a substantial non-perturbative component related to the well-known Fermi-motion effects [6].

The moments of the effective form factor,

$$f_N \equiv \int_0^1 dz\, z^{N-1}\, f(z), \tag{55}$$

can be written as the exponential of a series of functions:

$$f_N = \exp\left[L\; g_1\left(\beta_0 \alpha_S L\right) + g_2\left(\beta_0 \alpha_S L\right) + \alpha_S\, g_3\left(\beta_0 \alpha_S L\right) + \cdots\right], \tag{56}$$

where the leading and next-to-leading functions have the simple analytical expressions [10, 15, 7]:

$$\begin{aligned} g_1(\lambda) &= -\frac{A_1}{2\beta_0}\frac{1}{\lambda}\left[(1-2\lambda)\log(1-2\lambda) - 2(1-\lambda)\log(1-\lambda)\right]; \\ g_2(\lambda) &= \frac{\beta_0 A_2 - \beta_1 A_1}{2\beta_0^3}\left[\log(1-2\lambda) - 2\log(1-\lambda)\right] - \frac{\beta_1 A_1}{4\beta_0^3}\left[\log^2(1-2\lambda) - 2\log^2(1-\lambda)\right] + \\ &\quad + \frac{S_1}{2\beta_0}\log(1-2\lambda) + \frac{C_1}{\beta_0}\log(1-\lambda). \end{aligned} \tag{57}$$

We have defined $L \equiv \log n$ and $n \equiv N/N_0$, with $N_0 \equiv e^{-\gamma_E} = 0.561459\ldots$ and $\gamma_E = 0.577216\ldots$ the Euler constant. The first two coefficients of the β-function are:

$$\beta_0 = \frac{11 C_A - 2 n_F}{12\pi} = \frac{33 - 2 n_F}{12\pi}, \qquad \beta_1 = \frac{17 C_A^2 - 5 C_A n_F - 3 C_F n_F}{24\pi^2} = \frac{153 - 19\, n_F}{24\pi^2}, \tag{58}$$

where $C_A = N_c = 3$ and $n_F = 3$ is the number of active quark flavours. The one-loop quantities S_1 and C_1 have the values:

$$S_1 = -\frac{C_F}{\pi}, \qquad\qquad C_1 = -\frac{3}{4}\frac{C_F}{\pi}. \tag{59}$$

The two-loop quantity A_2 is given by [16, 17]:

$$A_2 = \frac{C_F}{2\pi^2} K \tag{60}$$

where, in the $\overline{MS}$ scheme for the coupling constant,

$$K = C_A\left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{10}{9} n_f T_R, \tag{61}$$

with $T_R = 1/2$.

The original form factor $f(z)$, in momentum space, is obtained with an inverse Mellin transform of f_N in (56):

$$f(z) = \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i}\, z^{-N}\, f_N, \tag{62}$$

where the constant c is chosen so that all the singularities of f_N lie to the left of the integration contour. The inverse transform is usually done numerically [18]. In leading order, the transform from f_N to the cumulative distribution

$$F(z) \equiv \int_z^1 f(z')\, dz' \tag{63}$$

is equivalent to the simple replacement:

$$\log n \to -\log y, \tag{64}$$

where

$$y \equiv 1 - z = \frac{m_X^2}{Q^2}. \tag{65}$$

The leading form factor can then be explicitly given in momentum space and reads:

$$f(z;\alpha_S)_l = \frac{d}{dy} \exp\left[h\left(y;\alpha_S\right)\right], \tag{66}$$

where

$$h\left(y;\alpha_S\right) \equiv -\frac{A_1}{2\beta_0^2\alpha_S}\left[\left(1+2\beta_0\alpha_S\log y\right)\log\left(1+2\beta_0\alpha_S\log y\right) - 2\left(1+\beta_0\alpha_S\log y\right)\log\left(1+\beta_0\alpha_S\log y\right)\right]. \tag{67}$$

It holds that $h\left(y=1\right)=0$. In agreement with the frozen coupling case (see later) we *define*: $\exp\left[h\left(y=0\right)\right]=0$. The QCD coupling is evaluated at the hard scale of the process:

$$\alpha_S \equiv \alpha_S\left(Q^2\right) = \alpha_S\left(w^2 m_B^2\right). \tag{68}$$

To some approximation, we can set $w=1$ in the running coupling so that

$$\alpha_S \simeq \alpha_S\left(m_B^2\right). \tag{69}$$

Equation (66) is simple but non-trivial, as the presence of the singularity for [6]

$$y \le y_{\min} \equiv \exp\left[-\frac{1}{2\beta_0\alpha_S}\right] \sim \frac{\Lambda}{Q} \tag{70}$$

shows. This singularity is related to the infrared pole in the running coupling. The frozen-coupling case is obtained by taking the limit $\beta_0 \to 0$ on the r.h.s of eq. (67) and reads:[4]

$$\begin{aligned} f(z,\alpha_S) &\approx \frac{d}{dy}\exp\left[-\frac{\alpha_S C_F}{2\pi}\log^2 y\right] \\ &= -\frac{\alpha_S C_F}{\pi}\frac{\log y}{y}\exp\left[-\frac{\alpha_S C_F}{2\pi}\log^2 y\right] \qquad (\beta_0=0). \end{aligned} \tag{71}$$

Our task is to factorize the long-distance contributions found in the hadronic tensor, i.e. the terms containing distributions. The simplest factorization scheme involves a minimal subtraction, so that the form factors are written:

$$W_i\left(\varsigma,w;\alpha_S\right) = V_i\left(w;\alpha_S\right)\, f(z;\alpha_S) + R_i\left(\varsigma,w;\alpha_S\right). \tag{72}$$

According to the above decomposition, the hadronic tensor is:

$$W_{\mu\nu}\left(\varsigma,w;\alpha_S\right) = V_{\mu\nu}\left(w;\alpha_S\right)\, f(z;\alpha_S) + R_{\mu\nu}\left(\varsigma,w;\alpha_S\right). \tag{73}$$

The tensors containing the virtual effects and the real ones are defined in a way analogous to the hadronic tensor:

$$\begin{aligned} V_{\mu\nu}\left(w;\alpha_S\right) &= \frac{1}{2v\cdot p_X}\left[\left(n_\mu v_\nu + n_\nu v_\mu - g_{\mu\nu}\, v\cdot n - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\right)V_1\left(w\right) - g_{\mu\nu}\,V_2\left(w\right) + \right. \\ &\qquad \left. + v_\mu v_\nu\,V_3\left(w\right) + \left(n_\mu v_\nu + n_\nu v_\mu\right)V_4\left(w\right) + n_\mu n_\nu V_5\left(w\right)\right], \end{aligned} \tag{74}$$

[4]The plus regularization of the function $\log\left(1-z\right)/\left(1-z\right)$ in the last member can be omitted because the exponential suppresses the virtual contribution for $\alpha_S \neq 0$.

and

$$R_{\mu\nu}\left(\varsigma,w;\alpha_S\right) = \frac{1}{2v\cdot p_X}\left[\left(n_\mu v_\nu + n_\nu v_\mu - g_{\mu\nu}\,v\cdot n - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\right)R_1\left(\varsigma,w\right) - g_{\mu\nu}\,R_2\left(\varsigma,w\right) + \right. \\ \left. + v_\mu v_\nu\,R_3\left(\varsigma,w\right) + \left(n_\mu v_\nu + n_\nu v_\mu\right)R_4\left(\varsigma,w\right) + n_\mu n_\nu R_5\left(\varsigma,w\right)\right]. \tag{75}$$

The form factors have an expansion in powers of α_S :

$$V_i\left(w;\alpha_S\right) = \delta_{i1} + \frac{\alpha_S C_F}{\pi}v_i\left(w\right) + \left(\frac{\alpha_S}{\pi}\right)^2 v_i'\left(w\right) + \cdots \tag{76}$$

and

$$R_i\left(\varsigma,w;\alpha_S\right) = -\delta_{i1}\delta\left(1-z\right)\theta\left(w-1\right) + \frac{\alpha_S C_F}{\pi}r_i\left(\varsigma,w\right) + \left(\frac{\alpha_S}{\pi}\right)^2 r_i'\left(\varsigma,w\right) + \cdots. \tag{77}$$

All the dependence on the heavy mass, i.e. on w, is contained into the short-distance form factors: the function $f(z;\alpha_S)$ depends only on the ratio of the final-state kinematical variables.

In leading approximation, the hadronic tensor has the particularly simple form:

$$W_{\mu\nu}\left(z,w;\alpha_S\right)_l = \frac{1}{m_B\,w}\left(n_\mu v_\nu + n_\nu v_\mu - g_{\mu\nu}\,v\cdot n - i\epsilon_{\mu\nu\alpha\beta}n^\alpha v^\beta\right)\,\frac{d}{dy}\exp\left[h\left(y;\alpha_S\right)\right], \tag{78}$$

with $h\left(y;\alpha_S\right)$ given by eq. (67).

2.3 Triple differential distribution

Let us now consider the most general distribution in (2), which is a triple differential distribution. One has basically to contract the hadronic tensor with the leptonic one. A third kinematical variable is involved, which we choose as the electron energy: $x \equiv x_e$. The expression in terms of the form factors reads:

$$\frac{1}{12\Gamma_0}\frac{d^3\Gamma}{dxdwdz}\left(z,w,x;\alpha_S\right) = \sum_{i=1}^{5}P_i\left(x,w,z\right)\,W_i\left(w,z;\alpha_S\right), \tag{79}$$

where $P_i\left(x,w,z\right)$ are polynomials in all the kinematical variables (independent of α_S):

$$\begin{aligned} P_1\left(x,w,z\right) &= \left[1+\overline{x}-w\right]\left[w-\overline{x}-\left(1-z\right)w^2\right]; \\ P_2\left(x,w,z\right) &= \frac{w}{2}\left[1-w+\left(1-z\right)w^2\right]; \\ P_3\left(x,w,z\right) &= \frac{w}{4}\left[\overline{x}\left(w-\overline{x}\right)-\left(1-z\right)w^2\right]; \\ P_4\left(x,w,z\right) &= \overline{x}\left(w-\overline{x}\right)-\left(1-z\right)w^2; \\ P_5\left(x,w,z\right) &= \frac{1}{w}\left[\overline{x}\left(w-\overline{x}\right)-\left(1-z\right)w^2\right]. \end{aligned} \tag{80}$$

Explicitly, one has:

$$\frac{1}{12\Gamma_0}\frac{d^3\Gamma}{dxdwdz}\left(z,w,x;\alpha_S\right) = \left[1+\overline{x}-w\right]\left[w-\overline{x}-\left(1-z\right)w^2\right]W_1 + \frac{w}{2}\left[1-w+\left(1-z\right)w^2\right]W_2 + \\ + \left[\overline{x}\left(w-\overline{x}\right)-\left(1-z\right)w^2\right]\left[\frac{w}{4}W_3 + W_4 + \frac{1}{w}W_5\right], \tag{81}$$

where $\overline{x} \equiv 1 - x$.

Let us now consider the kinematical constraints; there are various cases [3]. For a given electron energy, in the range

$$0 \leq \overline{x} \leq 1, \tag{82}$$

the range of the hadronic energy is

$$\overline{x} \leq w \leq 1 + \overline{x}, \tag{83}$$

and the range of y is:

$$\max\left(0, \frac{w-1}{w^2}\right) \leq y \leq \frac{\overline{x}\,(w - \overline{x})}{w^2}. \tag{84}$$

For a given hadronic energy w in the range

$$0 \leq w \leq 2, \tag{85}$$

the range of y is:

$$\max\left[0, \frac{w-1}{w^2}\right] \leq y, \tag{86}$$

and the range of the electron energy is:

$$\frac{w}{2}\left(1 - \sqrt{\varsigma}\right) \leq \overline{x} \leq \frac{w}{2}\left(1 + \sqrt{\varsigma}\right). \tag{87}$$

For a given hadronic mass ζ in the range

$$0 \leq \varsigma \leq 1, \tag{88}$$

the range of the hadronic energy is:

$$w \leq \frac{2}{1 + \sqrt{\varsigma}} \tag{89}$$

and the range of the electron energy is again given by eq. (87).

In the semi-inclusive region, the triple differential distribution is naturally written as:

$$\frac{1}{12\Gamma_0}\frac{d^3\Gamma}{dx\,dw\,dz}\left(z, w, x; \alpha_S\right) = C\left(w, x; \alpha_S\right)\, f\left(z; \alpha_S\right) + D\left(\varsigma, w, x; \alpha_S\right). \tag{90}$$

The coefficient function and the remainder function have a power series expansion in α_S :

$$\begin{aligned} C\left(w, x; \alpha_S\right) &= c_0\left(w, x\right) + \frac{\alpha_S C_F}{\pi}\, c\left(w, x\right) + \left(\frac{\alpha_S}{\pi}\right)^2 c'\left(w, x\right) + \cdots \\ D\left(z, w, x; \alpha_S\right) &= \frac{\alpha_S C_F}{\pi}\, d\left(z, w, x\right) + \left(\frac{\alpha_S}{\pi}\right)^2 d'\left(z, w, x\right) + \cdots . \end{aligned} \tag{91}$$

The function D has at most a logarithmic singularity $\sim \log^k(1-z)$ for $z \to 1$ and its lowest order vanishes: $d_0\left(z, w, x\right) = 0$. The explicit expression of the coefficient function reads:

$$C\left(w, x; \alpha_S\right) = \sum_{i=1}^{5} P_i\left(x, w, 1\right)\, V_i\left(w; \alpha_S\right) = P_1\left(x, w, 1\right) + \frac{\alpha_S C_F}{\pi} \sum_{i=1}^{5} P_i\left(x, w, 1\right)\, v_i\left(w\right) + \cdots . \tag{92}$$

Expanding in powers of α_S on both sides, the first two terms of the coefficient function read:

$$c_0(w,x) = P_1(x,w,1) = (1+\overline{x}-w)(w-\overline{x})\,; \tag{93}$$

$$\begin{aligned} c(w,x) &= \sum_{i=1}^{5} P_i(x,w,1)\; v_i(w) \\ &= (1+\overline{x}-w)(w-\overline{x})\left[-\frac{3}{2}\log w - \mathrm{Li}_2(1-w) - \frac{w\log w}{2(1-w)} - \frac{5}{4} - \frac{\pi^2}{3}\right] + \overline{x}(w-\overline{x})\frac{\log w}{2(1-w)}. \end{aligned} \tag{94}$$

The remainder function reads:

$$D(z,w,x;\alpha_S) = \sum_{i=1}^{5} P_i(x,w,z)\; R_i(w,z;\alpha_S) + \sum_{i=1}^{5}\left[P_i(x,w,z) - P_i(x,w,1)\right]\; V_i(w;\alpha_S)\; f(z;\alpha_S)\,. \tag{95}$$

Since

$$P_i(x,w,z) - P_i(x,w,1) = O(1-z)\,, \tag{96}$$

one can replace the fixed-order expansion of $f(z;\alpha_S)$ in the r.h.s. of eq. (95) and neglect the virtual contributions, i.e. the plus regularization. The one-loop contribution to the remainder function is:

$$d(z,w,x) = \sum_{i=1}^{5} P_i(x,w,z)\; r_i(w,z) + \left[P_1(x,w,1) - P_1(x,w,z)\right]\frac{\log(1-z)+7/4}{1-z}. \tag{97}$$

The explicit expression for d is quite long and we do not report it here.

Equation (90) is our main result and allows computing an arbitrary distribution in the threshold region to NLO logarithmic accuracy. It is the generalization to a triple differential distribution of the representation for the resummed shape variables for very small values of the resolution parameters, such as the thrust distribution for $1-T \ll 1$ [19].

In leading order, one needs the effective form factor $f(z;\alpha_S)$ in double-logarithmic approximation, eq. (66), and the coefficient function at the Born level given above, $c_0(w,x)$, in eq. (93). Replacing the expressions for the coefficient function and the effective form factor, the distribution reads, in leading order:

$$\begin{aligned} \frac{1}{\Gamma_0}\frac{d^3\Gamma}{dx\,dw\,dy} &= 12(1+\overline{x}-w)(w-\overline{x})\; f(z;\alpha_S) \\ &= 12(1+\overline{x}-w)(w-\overline{x})\frac{d}{dy}\exp\left[h(y;\alpha_S)\right], \end{aligned} \tag{98}$$

with $h(y)$ given in eq. (67).

In next-to-leading order (NLO), one needs $f(z;\alpha_S)$ to single-logarithmic accuracy and the one-loop functions $c(w,x)$ and $d(z,w,x)$. Since the term containing the long-distance effects depends only on z, the integrations over the electron energy and the hadronic energy do not touch the infrared logarithmic structure.

2.4 Distribution in the hadron and electron energy

Integrating over the variable y on both sides of eq. (98), in the range (84), one obtains for the leading distribution in the energies w and x:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dx\,dw} = 12(1+\overline{x}-w)(w-\overline{x})\left\{\exp\left[h\left(\frac{\overline{x}(w-\overline{x})}{w^2}\right)\right] - \theta(\Delta w)\exp\left[h\left(\frac{\Delta w}{w^2}\right)\right]\right\}, \tag{99}$$

where

$$\Delta w \equiv w - 1. \tag{100}$$

Since in the term proportional to $\theta(\Delta w)$, the hadron energy is restricted to the range $1 \leq w \leq 2$, according to the leading logarithmic approximation, we can set $w^2 = 1$. Introducing the neutrino energy $x_\nu \equiv 2E_\nu/m_B$ in place of the hadron energy, satisfying

$$x_e + x_\nu + w = 2, \tag{101}$$

the distribution can be written in the more "symmetrical" form:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dx_e dx_\nu} \simeq 12x_\nu(1-x_\nu)\left\{\exp\left[h\left(\frac{(1-x_e)(1-x_\nu)}{(2-x_e-x_\nu)^2}\right)\right] - \theta(1-x_e-x_\nu)\exp\left[h(1-x_e-x_\nu)\right]\right\}. \tag{102}$$

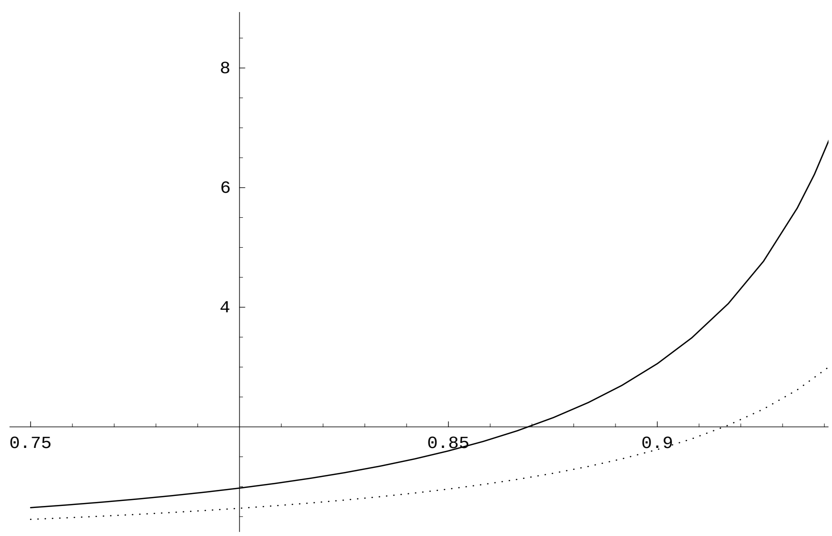

Fig. 1: Plot of the z distribution in leading logarithmic approximation in the range $0.75 \leq z \leq 0.95$. Solid line: running–coupling case; dotted line: frozen–coupling case.

Since we are interested to the semi-inclusive region $1 - x_e \ll 1$, we can set $x_e = 1$ whenever possible, so that:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dx_e dx_\nu} \simeq 12x_\nu(1-x_\nu)\left\{\exp\left[h\left(\frac{1-x_e}{1-x_\nu}\right)\right] - \theta(1-x_e-x_\nu)\exp\left[h(1-x_e-x_\nu)\right]\right\}. \tag{103}$$

In the frozen-coupling limit, the above expression reduces to:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dx_e dx_\nu} \simeq 12x_\nu(1-x_\nu)\left\{\exp\left[-\frac{\alpha_S C_F}{2\pi}\log^2\left(\frac{1-x_e}{1-x_\nu}\right)\right] - \theta(1-x_e-x_\nu)\exp\left[-\frac{\alpha_S C_F}{2\pi}\log^2(1-x_e-x_\nu)\right]\right\}. \tag{104}$$

The expansion to first order in α_S reads:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dx_e dx_\nu} \approx 12x_\nu(1-x_\nu)\left\{\theta(x_e+x_\nu-1) - \frac{\alpha_S C_F}{2\pi}\log^2\left(\frac{1-x_e}{1-x_\nu}\right) + \theta(1-x_e-x_\nu)\frac{\alpha_S C_F}{2\pi}\log^2(1-x_e-x_\nu)\right\}. \tag{105}$$

This formula contains the same double logarithms as the one-loop distribution computed in [3].

2.5 Distribution in the hadronic variables z and w

Performing the integration over the electron energy in the range (87), the double distribution in z and w reads, to leading logarithmic accuracy:

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma}{dwdy} = 2w^2\left(3-2w\right)\,f\left(y\right) = 2w^2\left(3-2w\right)\,\frac{d}{dy}\exp\left[h\left(y;\alpha_S\right)\right]. \tag{106}$$

Note the complete factorization in the variables w and y. Measuring this distribution, it is possible to determine in a direct way the shape function and to check the factorized structure.

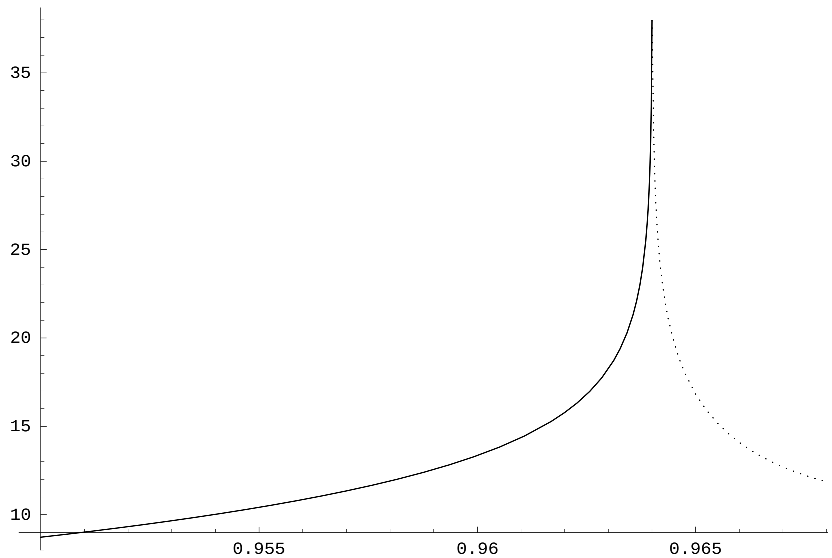

Fig. 2: Plot of the z distribution in the running coupling case for $z \geq 0.95$, i.e. for very large z. The dotted line represents the real part of the distribution, as the latter acquires an imaginary part after the peak.

2.6 Distribution in z

Integrating the above distribution over the hadronic energy in the range (89),

$$w \leq 1 + O\left(y\right), \tag{107}$$

one obtains the leading distribution in the (square of the) hadron mass normalized to the hadron energy:

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dy} = f\left(y\right) = \frac{d}{dy}\exp\left[h\left(y\right)\right]. \tag{108}$$

Note that this distribution does not coincide, beyond the double logarithm at one loop, with the hadron mass distribution, as discussed in the introduction (cf. eq. (29)). Since $f\left(y\right)$ is proportional to the shape function via a short-distance factor, a measure of this distribution allows a direct determination of the shape function. The distribution (108) is plotted in fig. 1. The region very close to $z=1$ is plotted in fig. 2. The maximum of $f\left(z\right)$ occurs at

$$z_{\max} \sim 1 - \frac{\Lambda}{Q}, \tag{109}$$

i.e. in the non-perturbative region described by the shape function[5]. The leading distribution (108) acquires indeed an imaginary part for $z > z_{\max}$.

[5] I wish to thank S. Catani for a discussion on this point.

2.7 Hadron energy spectrum

Integrating over y the distribution (106) according to condition (86) (the upper limit on y can be taken to be one for simplicity's sake), one obtains the resummed hadron energy spectrum in leading approximation:

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dw} = 2w^2\,(3-2w)\left\{1-\theta\,(w-1)\ \exp\left[h\,(w-1)\right]\right\}. \tag{110}$$

The above distribution acquires an imaginary part — and therefore is completely unphysical — in the region

$$0 \le \Delta w \le \exp\left[-\frac{1}{2\beta_0\,\alpha_S\,(m_B^2)}\right] \sim \frac{\Lambda}{m_B}. \tag{111}$$

This effect is related to the Landau pole in the coupling and implies that the region

$$E_X\,\in\,\left[\frac{m_B}{2},\,\frac{m_B}{2}+O\,(\Lambda)\right] \tag{112}$$

is non-perturbative. This region is described by the shape function because it is related to the integration over y down to $\Delta w \ll 1$.

In the frozen-coupling case, one has the simple expression:

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dw} \approx 2w^2\,(3-2w)\left\{1-\theta\,(w-1)\ \exp\left[-\frac{\alpha_S C_F}{2\pi}\log^2(w-1)\right]\right\} \qquad (\beta_0=0)\,. \tag{113}$$

The expansion to order α_S of the r.h.s. is in agreement with the double logarithmic approximation of the one-loop distribution computed in [3]:

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dw} = 2w^2\,(3-2w)\left\{\theta\,(1-w)+\theta\,(w-1)\ \left[\frac{1}{2}\frac{\alpha_S C_F}{\pi}\log^2(w-1)+\frac{7}{4}\frac{\alpha_S C_F}{\pi}\log\,(w-1)\right]+\cdots\right\}. \tag{114}$$

Note the factor 7/4 in front of the single logarithm, which is characteristic of the distributions not integrated over the hadron energy, as discussed in the introduction. The hadron spectrum is plotted in fig. 3. Let us remark that, in the frozen-coupling case, the distribution is smooth in the point $w=1$, because the exponential in eq. (113) goes to zero together with all its derivatives for $w\to+1$. A plot of a small neighborhood of this point would indeed show that the apperent cusp in fig. 3 is actually not there.

3 Rare decay

In this section we present an improved expression for the photon spectrum near the endpoint in the rare decay (25) [20, 21, 22]. The only terms in the spectrum containing large infrared logarithms to $O\,(\alpha_S)$ involve two insertions of the operator O_7 (see [22] for the definition of the operator basis). The latter is also the dominant one in the inclusive rate, so this operator usually is the only one considered. It is however possible to improve the distribution by including the other operators, as we are going to show.

The hadronic variables z and w introduced to describe the semileptonic decay are redundant in this case, because the photon has $q^2=0$; we have:

$$\begin{aligned} 1-z &= \frac{1-x}{(2-x)^2}\approx 1-x, \\ w &= 2\,-\,x\ \approx\ 1, \end{aligned} \tag{115}$$

where $x \equiv x_\gamma$. The variables z and x basically coincide in the semi-inclusive region while the hadron energy is fixed, as anticipated in the introduction. We therefore consider the x spectrum. The improved distribution we propose reads:

$$\frac{d\Gamma}{dx} = \frac{G_F^2 \alpha_{em}}{32\pi^4} |V_{tb} V_{ts}^*|^2 m_{b,pole}^3 m_{b,\overline{MS}}^2 (m_b) \left[Q(\mu_b) \, f(x) - \rho'(x) \right] + O\left(\alpha_S^2\right), \tag{116}$$

where

$$Q(\mu_b) \equiv |\widetilde{C}_7^{(0)}(\mu_b)|^2 + \frac{\alpha_S(\mu_b)}{2\pi} \mathrm{Re} \left\{ \widetilde{C}_7^{(0)}(\mu_b)^* \left[\widetilde{C}_7^{(1)}(\mu_b) + \sum_{i=1}^{8} \widetilde{C}_i^{(0)}(\mu_b) \left(r_i + \widetilde{\gamma}_{i7}^{(0)} \log \frac{m_b}{\mu_b} \right) \right] \right\}. \tag{117}$$

The "remainder" function $\rho'(x)$ is the derivative of:

$$\rho(x) \equiv \frac{\alpha_S(\mu_b)}{\pi} \sum_{i \le j}^{1,8} \widetilde{C}_i^{(0)}(\mu_b) \, \widetilde{C}_j^{(0)}(\mu_b) \, f_{ij}(1-x). \tag{118}$$

The latter vanishes at the endpoint:

$$\rho(x) \to 0 \qquad \text{for} \qquad x \to 1, \tag{119}$$

because the functions $f_{ij}(1-x)$ all vanish in the endpoint:

$$f_{ij}(1-x) \to 0 \qquad \text{for} \qquad x \to 1. \tag{120}$$

The function $\rho'(x)$ has at most a logarithmic singularity $\sim \log(1-x)$ for $x \to 1$. The quantities $\widetilde{C}_i^{(0)}$ and $\widetilde{C}_i^{(1)}$ are the leading and next-to-leading contributions to the effective coefficient functions:

$$\widetilde{C}_i(\mu) = \widetilde{C}_i^{(0)}(\mu) + \frac{\alpha_S(\mu)}{4\pi} \widetilde{C}_i^{(1)}(\mu) + \cdots. \tag{121}$$

r_i are complex constants depending on the ratio m_c/m_b. The definition of all the symbols can be found in [22].

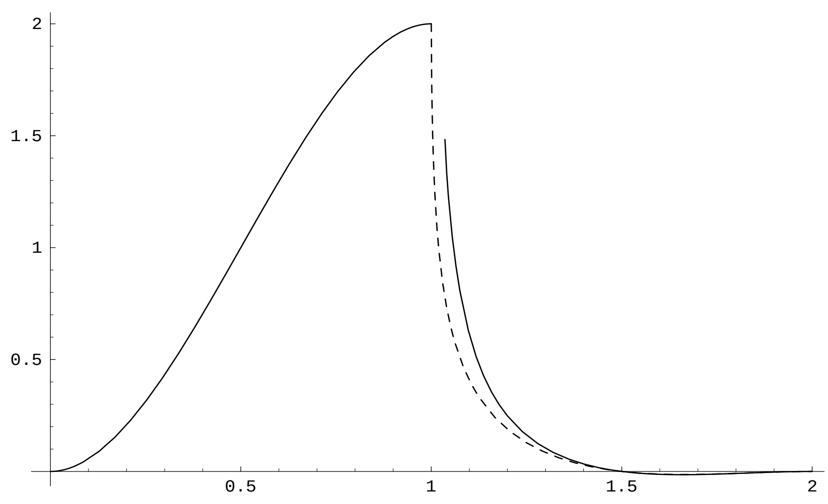

Fig. 3: Plot of the energy spectrum in leading logarithmic approximation. For $w > 1$, the dashed line corresponds to frozen coupling, the continuous line to running coupling. In the latter case, the distribution is not well defined close to the point $w = 1$ because of Landau-pole effects.

To avoid large logarithms in the matrix elements, one has to take:

$$\mu_b = O\left(m_b\right). \tag{122}$$

The long-distance effects are contained in the function $f\left(x\right)$,which receives all the non-perturbative contributions (in leading twist) related to Fermi motion. The main point is that $f\left(x\right)$ is the same function $f\left(z\right)$, which was introduced to factorize the long-distance effects in the semileptonic decay (cf. eq. (90)). Therefore we explicitly verify the universality of the long-distance contributions in (1), once they are factorized by means of a function of z. The function $\rho'\left(x\right)$ is instead a short-distance contribution, specific of the process (25).

The proof of eq. (116) is trivial. The integrated distribution in the upper part of the photon spectrum,

$$\Gamma_{up}\left(x\right) \equiv \int_x^1 \frac{d\Gamma}{dx'}dx', \tag{123}$$

given in eq. (30) of [22], can be written, up to terms of order α_S^2, as:

$$\begin{aligned} \Gamma_{up}\left(x\right) \quad = \quad & \frac{G_F^2\alpha_{em}}{32\pi^4} m_{b,pole}^3 m_{b,\overline{MS}}^2\left(m_b\right)\left|V_{tb}V_{ts}^*\right|^2 \\ & \left\{Q\left(\mu_b\right)\left[\theta\left(1-x\right) - \frac{1}{2}A_1\alpha_S\left(\mu_b\right)\log^2\left(1-x\right) + B_1\alpha_S\left(\mu_b\right)\log\left(1-x\right)\right] + \rho\left(x\right)\right\} + O\left(\alpha_S^2\right). \end{aligned} \tag{124}$$

One then takes a derivative with respect to x. The requirement that the spectrum is non-singular at $x = 1$ — or equivalently, that the total rate is correctly reproduced[6] — transforms the distributions in plus distributions and one obtains eq. (116).

4 Conclusions

We have performed factorization and threshold resummation of the most general distribution in the semileptonic $b \to u$ decay. This has been achieved with a proper choice of the kinematic variables,

$$w \equiv \frac{2E_X}{m_B} \qquad \text{and} \qquad z \equiv 1 - \frac{m_X^2}{4E_X^2}, \tag{125}$$

which disentangle the two different logarithmic structures occurring in the process:

$$1): \ \log w \qquad \text{and} \qquad 2): \ \log\left(1-z\right). \tag{126}$$

The first structure has been related to the infinite mass limit of the b quark,

$$m_B \to \infty, \tag{127}$$

while the second one has been related to the infinite energy limit of the final hadronic system,

$$E_X \to \infty. \tag{128}$$

The long-distance effects — both perturbative and non-perturbative — have been relegated to a universal factor $f\left(z\right)$, depending only on z. This function is related to the shape function $\varphi\left(k_+\right)$ by a short-distance coefficient function. In essence, in our approach, the hard scale of the process is the hadronic energy E_X in the B rest frame, and not the B meson mass m_B.

We have presented simple analytical expressions for a few distributions, resummed to leading logarithmic accuracy and we have shown that the shape function can be directly determined by measuring the distribution in z or in z and w.

[6] We do not consider here the problem of the infrared singularities for $x \to 0$, related to soft-photon emission.

We have computed the resummed hadron energy spectrum, which exhibits a "Sudakov shoulder" in the point

$$E_X = \frac{m_B}{2}, \tag{129}$$

i.e. in the middle of the allowed kinematical domain $(0 \le E_X \le m_B)$. The spectrum very close to this point is non-perturbative and is proportional to the shape function. This implies that, at least in principle, an accurate measure in this region may lead to another independent determination of the shape function. It is a non-trivial fact that the same long-distance effects appear at the boundary of the phase space in the z-distribution while they appear inside the allowed kinematical domain in the energy spectrum. The computation of other resummed distributions in LO or NLO by integrating eq. (90) is straightforward. A crucial point in our analysis is that there is a single source of large logarithms in any semi-inclusive distribution in heavy-flavour decay.

Finally, we have presented an improved formula for the photon spectrum in the rare decay (25), which takes into account soft-gluon resummation and the effects of the subleading operators. It has been explicitly shown that the same function $f(z)$ factorizes the long-distance effects in the semileptonic and in the rare decay.

We believe that our formalism sets a rather general and rigorous scheme for the separation of perturbative from non-perturbative effects in semi-inclusive heavy-flavour decays and allows for a simple analysis of the experimental data of many different distributions.

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